MATH 245 F22, Exam 3 Solutions

1. Carefully define the following terms: subset, symmetric difference
Given two sets A, B, we say that A is a subset of B if every element of A is an element of B. Given two sets A, B, we define their symmetric difference as the *set* $\{x : (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}$.

2. Carefully define the following terms: reflexive, reflexive closure
Given R, a relation on set S, we say that R is reflexive if for every $x \in S$, we have $(x, x) \in R$. Given R, a relation on set S, we define the reflexive closure of R as $R \cup \{(x, x) : x \in S\}$.

3. Prove or disprove: For all sets A, B, C which satisfy $A \subseteq C \subseteq B$, we must have $A \times B \subseteq B \times C$.
The statement is false. We now need to find specific sets A, B, C with $A \subseteq C \subseteq B$ (and $C \neq B$), and find a specific element of $A \times B$ that is not in $B \times C$.

   Many solutions are possible. For example, take $A = C = \{1\}$ and $B = \{1, 2\}$. This satisfies $A \subseteq C \subseteq B$, and the element $(1, 2) \in A \times B$ but $(1, 2) \notin B \times C$.

4. Let $S = \{x \in \mathbb{Z} : 81|x^3\}$ and $T = \{x \in \mathbb{Z} : 3|x^3\}$. Prove or disprove that $S = T$.
The statement is false. To disprove requires finding some specific $x \in T$ with $x \notin S$ (in fact, $S \subseteq T$, so the other way won't work).

   The simplest counterexample is 3. Note that $3|3^3$ because $3 \times 9 = 27 = 3^3$; hence $3 \in T$. However, $81 \nmid 3^3$, since the only solution to $81n = 27$ is $n = \frac{1}{3}$, which is not an integer. Hence $3 \notin S$.

5. Let A, B be sets with $A \setminus B \subseteq B \setminus A$. Prove that $A \subseteq B$.
Let $x \in A$ be arbitrary. Our task is to prove $x \in B$.

   METHOD 1: We now have two cases, based on whether or not $x \in B$.
   Case $x \in B$: We are happy, since this was our goal.
   Case $x \notin B$: By conjunction, $x \in A \wedge x \notin B$. Hence $x \in A \setminus B$. By hypothesis, $x \in B \setminus A$. Hence, $x \in B \wedge x \notin A$. By simplification, $x \in B$.
   In both cases, $x \in B$.

   METHOD 2: We will prove $x \in B$ by contradiction. Suppose otherwise, i.e. $x \notin B$. By conjunction, $x \in A \wedge x \notin B$. Hence $x \in A \setminus B$. By hypothesis, $x \in B \setminus A$. Hence, $x \in B \wedge x \notin A$. By simplification, $x \notin A$. This contradicts $x \in A$. Hence $x \in B$.

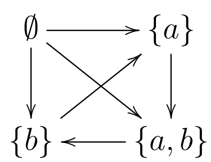
6. Consider the relation $R_{diagonal}$ on $\mathbb{N}$. Prove or disprove that $|\mathbb{N}| = |R_{diagonal}|$.
The statement is true. To prove these two infinite sets are equicardinal requires a pairing. Fortunately, there is a natural one available: $1 \leftrightarrow (1, 1), 2 \leftrightarrow (2, 2), 3 \leftrightarrow (3, 3), \ldots$.
Or, more generally, $n \leftrightarrow (n, n)$.

7. Let S, U be sets with $S \subseteq U$. Prove that $(S^c)^c \subseteq S$.
Note: This is part of Theorem 9.2. Do not use this theorem to prove itself!

Let $x \in (S^c)^c$. Then $x \in U \setminus S^c$, so $x \in U \wedge x \notin S^c$ or $x \in U \wedge \neg(x \in S^c)$. But now $x \in U \wedge \neg(x \in U \wedge x \notin S)$. Applying De Morgan's Law (for propositions) we get $x \in U \wedge (x \notin U \vee x \in S)$. Finally, we apply Disjunctive Syllogism to get $x \in S$.

8. Let $S = \{a, b\}$, and $T = 2^S$. Find a trichotomous relation on T. Give your answer both as a digraph and as a set.

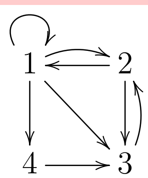
Note that there are six ways to pick different $x, y \in T$, so your relation must contain at least six edges. Many solutions are possible; here is one:



$R = \{(\emptyset, \{a\}), (\emptyset, \{a, b\}), (\emptyset, \{b\}), (\{a\}, \{a, b\}), (\{b\}, \{a\}), (\{a, b\}, \{b\})\}$.
Notation is important here: we need sets, inside of ordered pairs, inside of a set.

For problems 9,10:
Let $A = \{1, 2, 3, 4\}$ and take $R = \{(a, b) : a|(b + 1)\}$, a relation on A.

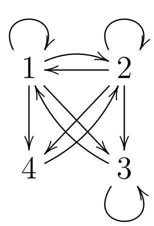
9. Draw the digraph representing R. Determine, with justification, whether or not R is each of: symmetric, antisymmetric, and transitive.



R is not symmetric because, e.g., $(1, 3) \in R$ and $(3, 1) \notin R$.
R is not antisymmetric because, e.g. $(1, 2), (2, 1) \in R$.
R is not transitive because, e.g., $(2, 1), (1, 4) \in R$ and $(2, 4) \notin R$.

Even if you have the wrong digraph, you can potentially get points for determining the three properties, provided you do so correctly for *your* digraph.

10. Compute $R \circ R$. Give your answer as a digraph.



Note: Every missing or extra edge will cost points.
It is possible to get full credit while not getting Problem 9 completely right. However, if your errors in Problem 9 substantially change the answer to Problem 10, then it is no longer possible to get full credit to Problem 10 since you are no longer demonstrating all the skills being tested.